UNITARY INVARIANCE OF THE KOSTLAN NORM (LINEAR ALGEBRA PROOF)

GREGORIO MALAJOVICH

Let $\mathcal{H}_d$ be the space of systems of n homogeneous polynomials of degree $d = (d_1, \dots, d_n)$ in variables $x_0, \dots, x_n$, with complex coefficients. The *Kostlan norm* $\|.\|_{\mathrm{k}}$ is defined in $\mathcal{H}_d$ as follows :

$$\|F\|_{\mathrm{k}}{}^2 = \sum_i \|F_i\|_{\mathrm{k}}{}^2$$

where, for any single polynomial f of degree d in variables $x_0, \dots, x_n$, we set :

$$\|f\|_{\mathrm{k}}{}^2 = \sum_{|J|=d} \frac{|f_J|^2}{\begin{pmatrix} d \\ J \end{pmatrix}}$$

Above, J are multi-indices, and :

$$\begin{pmatrix} d \\ J \end{pmatrix} = \frac{d!}{J_0! J_1! \dots J_n!}$$

We will give an easy proof of the Theorem :

Theorem 1 (Kostlan). *Let $F \in \mathcal{H}_d$. Then for any unitary automorphism U of $\mathbb{C}^{\ltimes+\nVdash}$, $\|F\|_k = \|F \circ U\|_k$.*

This theorem was proven by Eric Kostlan in [2]. I do not have his paper, but I took a look once, and (if I remember correctly) he gave a diferent proof.

Unitary invariance of the Kostlan norm was extremely important to the development of the complexity theory of solving systems of multivariate polynomials (see e.g. Mike Shub and Steve Smale [3]), since $\|.\|_{\mathrm{k}}$ is the natural norm in $\mathcal{H}_d$.

We first prove the Lemma :

Lemma 1. *Let $f = \sum f_j x^{i-j} y^j$ be a homogeneous polynomial of degree i in x and y, with coefficients f_j in a complex vector space K with*

Date: February 4, 1994.

Hermitian inner product $\langle .,. \rangle$. *Let* $\left\langle f, \tilde{f} \right\rangle_k$ *denote :*

$$\left\langle f, \tilde{f} \right\rangle_k = \sum_{0 \leq j \leq i} \frac{\left\langle f_j, \tilde{f}_j \right\rangle}{\begin{pmatrix} i \\ j \end{pmatrix}}$$

Then for any $U \in U(2)$, $\|f\|_k = \langle f, f \rangle_k^{\frac{1}{2}} = \langle f \circ U, f \circ U \rangle_k^{\frac{1}{2}} = \|f \circ U\|_k$.

The Lemma implies the Theorem for $n = 2$, by making $K = \mathbb{C}$ and $\langle w, z \rangle = w\bar{z}$. We first prove the Lemma. Then we will show that the Lemma implies the Theorem in general.

Proof of the Lemma :

We first show that $\|.\|_{\mathrm{k}}$ is invariant under ordinary rotations of the form :

$$U_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Indeed, let $f_t = f \circ U_t$. It suffices to prove that for any f, we have :

$$\mathrm{Re}\left(\left\langle f, \dot{f} \right\rangle_{\mathrm{k}} \right) = 0$$

where $\dot{f}$ means $\frac{\partial f_t}{\partial t}$ when $t = 0$. Indeed, this will show that for any t,

$$\frac{\partial}{\partial t} \|f_t\|_{\mathrm{k}} = 2 \ \mathrm{Re}\left(\left\langle f, \dot{f} \right\rangle_{\mathrm{k}} \right) = 0$$

We write :

$$\dot{f}(0) = \sum_{0 \leq j < i} f_j (i - j) x^{i-j-1} y^{j+1} \ - \sum_{0 < j \leq i} f_j j x^{i-j+1} y^{j-1}$$

Therefore,

$$\left\langle f, \dot{f} \right\rangle_{\mathrm{k}} = \sum_{0 \leq j < i} \frac{\langle f_{j+1}, f_j \rangle (i - j)}{\begin{pmatrix} i \\ j+1 \end{pmatrix}} \ - \sum_{0 < j \leq i} \frac{\langle f_{j-1}, f_j \rangle j}{\begin{pmatrix} i \\ j-1 \end{pmatrix}}$$

We substitute j by $j + 1$ in the first sum and get :

$$\left\langle f, \dot{f} \right\rangle_{\mathrm{k}} = \sum_{0 < j \leq i} \frac{\langle f_j, f_{j-1} \rangle (i - j + 1)}{\begin{pmatrix} i \\ j \end{pmatrix}} \ - \sum_{0 < j \leq i} \frac{\langle f_{j-1}, f_j \rangle j}{\begin{pmatrix} i \\ j-1 \end{pmatrix}}$$

$$\left\langle f, \dot{f} \right\rangle_{\mathrm{k}} = 2i\mathrm{Im}\left(\sum_{0 < j \leq i} \frac{j! i - j + 1!}{i!} \langle f_j, f_{j-1} \rangle \right)$$

Therefore, $\left\langle f, \dot{f} \right\rangle_{\mathrm{k}}$ is a pure imaginary, and $\|f_t\|_{\mathrm{k}}$ does not depend on t. This proves that $\|.\|_{\mathrm{k}}$ is rotation-invariant. A general unitary transformation can be represented a rotation times a complex in the unit circle, times (eventually) permutation of variables. This proves the Lemma.

Proof of the Theorem :

We first prove that $\|.\|_{\mathrm{k}}$ is invariant under (complex) Givens rotations. A Givens rotation is a rotation in a plane generated by two complex coordinates, and leaving all other coordinates invariant. Without loss of generality, we can show that fact for rotations of variables x_0 and x_1 only. We write f as :

$$f(x) = \sum_{0 \le i \le d} \sum_{0 \le j \le i} {x_0}^{i-j} {x_1}^{j} f_{ij}(x_2, \ldots, x_n)$$

It is easy to see that :

$$\left\langle f, \tilde{f} \right\rangle_{\mathrm{k}} = \sum_{0 \le i \le d} \sum_{0 \le j \le i} \frac{\left\langle f_{ij}, \tilde{f}_{ij} \right\rangle_{\mathrm{k}}}{\frac{d!}{d-i!i-j!j!}} = \sum_{0 \le i \le d} \sum_{0 \le j \le i} \frac{\left\langle f_{ij}, \tilde{f}_{ij} \right\rangle_{\mathrm{k}}}{\begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} d \\ i \end{pmatrix}}$$

According to the Lemma, for each i, the real part of the sum in j is invariant by unitary transforms of x_0 and x_1. Therefore, $\mathrm{Re}(\langle ., . \rangle_{\mathrm{k}})$ is invariant by unitary transforms of x_0 and x_1. Hence, $\|.\|_{\mathrm{k}}$ is invariant under (complex) Givens rotations. We can use the well-known :

Proposition 1. *Every element of $U(n+1)$ can be written as a product of (complex) Givens rotations* and (eventually) a variable permutation.

It follows from the proposition that $\|f\|_{\mathrm{k}}$ is $U(n+1)$ -invariant, and the invariance Theorem is proved.

The proof of this proposition uses the same argument than the Givens QR factorization. Namely, let A be a unitary transform. By applying a variable permutation, we can assume that $detA = +1$. Now we reproduce the classical argument (See Golub and Van Loan [1], Algorithm 5.2.2 page 214 or Watkins [4], Theorem 3.2.9, page 143) :

We assumed that $A^{(0)} = A \in SU(n+1)$. There is a Givens rotation $G^{(0)}$ changing only coordinates n and $n+1$ such that the matrix $A^{(1)} = G^{(0)} A^{(0)}$ verifies :

$$A^{(1)}{}_{n+1,1} = 0$$

Namely, we set :

$$c = \frac{\bar{A^{(0)}}_{n+1,1}}{\sqrt{|A^{(0)}{}_{n,1}|^2 + |A^{(0)}{}_{n+1,1}|^2}}$$

$$s = \frac{\bar{A}^{(0)}{}_{n,1}}{\sqrt{\left|A^{(0)}{}_{n,1}\right|^2 + \left|A^{(0)}{}_{n+1,1}\right|^2}}$$

$$G^{(0)} = \begin{pmatrix} I & & \\ & c & s \\ & -s & c \end{pmatrix}$$

By the same procedure, we construct

$$G^{(1)} = \begin{pmatrix} I & & & \\ & c & s & \\ & -s & c & \\ & & & 1 \end{pmatrix}$$

such that $A^{(1)} = G^{(1)} A^{(0)}$ verifies :

$$A^{(1)}{}_{n,1} = A^{(1)}{}_{n+1,1} = 0$$

and so on. We will then construct $A^{(n)} = \prod_{i=n-1}^{0} G^{(i)}\ A^{(0)}$ such that $A^{(n)}{}_{i,1} = 0$ for $i > 1$ and $A^{(n)}{}_{1,1} = 1$. We can then construct a Givens rotation $G^{(n)}$,

$$G^{(n)} = \begin{pmatrix} I & & \\ & c & s \\ & -s & c \end{pmatrix}$$

such that $A^{(n+1)} = G^{(n)} A^{(n)}$ verifies $A^{(n+1)}{}_{n+1,2} = 0$, $A^{(n+1)}{}_{i,1} = 0$ for $i > 1$ and $A^{(n+1)}{}_{1,1} = 1$. By the same procedure as above, we obtain $A^{(2n-1)}$ with all $A^{(2n-1)}{}_{i,j} = 0$ for $i > j$ and $j = 1, 2$, and $A^{(2n-1)}{}_{i,i} = 1$ for $i = 1, 2$.

Continuing the same procedure, we finally get a matrix

$$A^{\left(\frac{n(n-1)}{2}\right)} = \prod_{i=\frac{n(n-1)}{2}-1}^{0} G^{(i)}\ A^{(0)}$$

that is upper triangular and has only 1's in the diagonal. Since $A^{\left(\frac{n(n-1)}{2}\right)}$ is also a rotation (an element of $SU(n+1)$), it follows that it is precisely the identity of $\mathbb{C}^{n+1}$.

Therefore, we can write :

$$A^{(0)} = \prod_{i=0}^{\frac{n(n-1)}{2}-1} {G^{(i)}}^{-1}$$

and A is a product of Givens rotations. This concludes the proof of the Proposition.

References

[1] Gene Golub and Charles Van Loan, *Matrix Computations*, Second Edition. The John Hopkins University Press, Baltimore and London, 1990.

[2] Eric Kostlan, *Random polynomials and the statistical fundamental theorem of algebra*, Preprint, Univ. of Hawaii, 1987.

[3] Michael Shub and Steve Smale, On the Complexity of Bezout's Theorem I - Geometric aspects. *Journal of the AMS*, **6**, 2, Apr 1993.

[4] David S. Watkins, *Fundamentals of matrix computations*, John Wiley & Sons, New York, 1991

Departamento de Matematica Aplicada da UFRJ, Caixa Postal 68530, CEP 21945, Rio de Janeiro, RJ, BRASIL

E-mail address: gregorio@labma.ufrj.br